Exam. Code : 103204
Subject Code : 1126

B.A./B.Sc. 4th Semester

MATHEMATICS

Paper—II

(Number Theory)

Time Allowed—3 Hours] [Maximum Marks—50

Note :— Attempt any **FIVE** questions, selecting at least **TWO** questions from each section. All questions carry equal marks.

SECTION—A

1. (a) Prove that if a and b are integers, with $b > 0$, then there exist unique integers q and r satisfying $a = qb + r$, where $2b \leq r < 3b$.

   (b) By Division Algorithm of $n \geq 1$, prove that $n(n + 1)(n + 2)/6$ is an integer.

2. (a) Prove that if a and b are given integers, not both zero, then the set

   $T = \{ax + by \mid x, y \text{ are integers}\}$ is precisely the set of all multiples of $d = \gcd(a, b)$.

   (b) Prove that if a and b are both odd integers, then $16/a^4 + b^4 - 2$.

3. (a) Prove that if d is a common divisor of a and b, then $d = \gcd(a, b)$ if and only if $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$.

(b) Determine all solutions in the positive integers of Diophantine equation $18x + 5y = 48$.

4. (a) Prove that only prime of the form $n^3 - 1$ is 7.

(b) If p_n is the nth prime number, prove that $p_n \leq 2^{2^{n-1}}$.

5. (a) Prove that an integer is divisible by 4 if and only if the number formed by its ten units digits is divisible by 4.

(b) Use the theory of congruencies to verify $89/2^{44} - 1$.

SECTION—B

6. State and prove Chinese remainder theorem and solve the following set of simultaneous congruencies $x \equiv 1 \pmod 3$, $x \equiv 2 \pmod 5$, $x \equiv 3 \pmod 7$.

7. (a) Employ Fermat's Theorem to prove that, if p is a odd prime, then

$$1^{p-1} + 2^{p-1} + 3^{p-1} + \ldots\ldots (p-1)^{p-1} \equiv -1 \pmod p.$$

(b) Using Wilson's Theorem, prove that

$$1^2 . 3^2 . 5^2 \ldots\ldots (p-2)^2 \equiv (-1)^{(p+1)/2} \pmod p.$$

8. (a) Prove that the functions π and σ are both multiplicative functions.

(b) Prove that for each positive integer $n \geq 1$, $\sum_{d/n} \mu(d) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n > 1 \end{cases}$ where d runs through the positive divisors of n.

9. (a) Show that 1000 ! terminates in 249 zeros.

(b) State and prove Euler's Theorem.

10. (a) Prove that for $n > 1$, the sum of the positive integers less than n and relatively prime to n is $\frac{1}{2} n\varphi(n)$.

(b) Show that for any integer $n \geq 3$, $\sum_{k=1}^{n} \mu(k!) = 1$.